computations show that this condition is fulfilled for $M_c \leq 1.3$. The fields of density and methane mass fraction obtained after ignition and stabilization of combustion are plotted in Fig. 3. It can be observed that combustion results in an increase in the size of the recirculation bubble and in an upstream displacement of the leading shock of about 7 cm. These features could be due to the decrease of the density and the dynamic turbulent viscosity $\mu_t = \rho\nu_t$ and thus to the corresponding change of the force balance acting on the bubble.

IV. Conclusion

In this work, a technique to stabilize combustion within supersonic flows using a free recirculating bubble has been presented. Although both the turbulence model and chemical kinetics mechanism used here remain extremely simple, the overall features of the flow are expected to stay unchanged when these approximations are refined for the sake of quantitative analysis. It is clear that the computations performed here are unable to predict unsteady properties of the system, such as vortex shedding or oscillations of the bubble, that can play a significant role in the ignition and stabilization capabilities of this system. Calculations to determine precisely the role of different flow parameters on the stability limits of the free recirculation bubble with combustion are now being performed. The preliminary results presented here need to be extended to the case of practical interest of a swirling jet.

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Some Practical Complete Modal Spaces and Equivalence

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Introduction

THE modal expansion method has been found to be very useful for modal synthesis, calculation of eigenvector derivatives, model correction, and reduction of dynamic models. Because there exists numerical error due to truncation of modes, the precision of the results sometimes is poor. To reduce the error of truncated modes, a practical complete modal space (PCMS) is developed. This Note briefly describes successful applications of the PCMS in some practical engineering areas. To satisfy the requirement in different applications, the PCMS has been further improved.

In the PCMS method, some lower-order modes including rigid-body modes are obtained from solving the eigenequation. All higher-order modes are replaced by the equivalent higher-order modes that are given by using a simple matrix projection approach. This replacement means the entire contribution of the equivalent higher-order modes is equivalent to that of the original higher-order modes. Also the subspace spanned by the equivalent higher-order modes is directly equivalent to the subspace spanned by original higher-order modes. The direct proof of equivalence between two subspaces spanned by both the equivalent higher-order modes and original higher-order modes is discussed in this Note.

With respect to the modal synthesis, the existing free-interface method,[1] the fixed-interface method,[2] and the mixed-interface method[3] are all approximately substructural coupling methods based on an incomplete modal space. The accurate modal synthesis methods[4–6] based on the PCMS method can give better precision of any order of modes for assembly structure because the precision of the Rize analysis is determined principally by the completeness of the basis of the vector space. In the calculation of eigenvector derivatives, Fox and Kapoor[7] developed an incomplete modal expansion technique. Wang[8] improved the Fox–Kapoor method by adding a static correction term to the modal expansion formula. But their methods merely guarantee that the precision of the eigenvector derivatives of a few lower-order modes is good. Otherwise a complete modal method[9] based on the PCMS theory can make the precision of the derivatives of many higher-order modes accurate. In regard to the reduction of a dynamic model, Kammer[10] proposed an incomplete modal method that can only guarantee that some lower-order modes of the reduced model are exact inside the frequency range of interest. Reference 11 makes available many modes of the reduced model outside the range of interest. In the field of model correction, Berman et al.,[12] Berman and Wei,[13] Zhang and Li,[14] and Kabe[15] developed various methods of model correction, which are based on an incomplete modal space. A complete mode-type method was described in Ref. 16. In the present method, the PCMS formed by lower-order measured modes and equivalent higher-order modes is arranged as the reference base. Based on this reference base, there are matrix-type, element-type, submatrix-type, and design-parameter-type methods. The results from the matrix-type and element-type methods based on the PCMS show that the

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lower-order and higher-order eigenpairs of the modified model with full matrices are the same as the measured and analytic eigenpairs, respectively.[16] This characteristic is included in the updated model with band-state matrices.

The orthogonality between the column vectors of the equivalent higher-order modes with respect to the mass matrix M and stiffness matrix K does not exist. This will reduce the computational efficiency when μ_h^{-1} and $\mathcal{K}_h^{-1}$ are used in the computation. In addition, this nonorthogonality behavior may sometimes reduce the precision for certain applications[11] of PCMS theory. To make the equivalent higher-order modes that are orthogonal to M and K, a triangular decomposition technique and an orthogonalization method are presented in this paper. The orthogonalization method is based on an improved form (called the optimal Gram–Schmidt orthogonalization method) of the method mentioned in Ref. 17 and can make the equivalent higher-order modes be simultaneously orthogonal to both M and K.

Practical Complete Modal Spaces and Equivalence

The eigenequation of a system with n degrees of freedom is

$$K\varphi_i = \lambda_i M\varphi_i, \qquad i = 1, 2, \ldots, n \tag{1}$$

in which M is a positive definite mass matrix and K is a positive semidefinite stiffness matrix. The terms $\Lambda_k = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_k]$ and $\Phi_k = [\varphi_1, \varphi_2, \ldots, \varphi_k] \in R^{n,k}$ represent k numbers of lower-order eigenpairs that are computed from Eq. (1) and satisfy $\Phi_k^T M \Phi_k = I$ and $\Phi_k^T K \Phi_k = \Lambda_k$. Thus, it is known that

$$M^{-1} = \Phi_k \Phi_k^T + \Phi_h \Phi_h^T \tag{2}$$

in which $\Phi_h = [\varphi_{k+1}, \ldots, \varphi_n] \in R^{n,h} (h = n - k)$ are mass-normalized higher-order modes of Eq. (1). This is to say that $\Phi_h^T M \Phi_h = I$ and $\Phi_h^T K \Phi_h = \Lambda_h$. Note that Φ_h and $\Lambda_h = \text{diag}[\lambda_{k+1}, \ldots, \lambda_n]$ do not need to be calculated from Eq. (1). From Eq. (2) we have[18]

$$\Phi_h \Phi_h^T M = I - \Phi_k \Phi_k^T M = \bar{\Psi}_h \ (\in R^{n,n}) \tag{3}$$

The rank of matrix $(\Phi_h \Phi_h^T)$ is h, and the rank of M is n; therefore, the rank of matrix $\bar{\Psi}_h$ equals h, where $h = n - k$. This denotes that there are only h independent columns in the matrix $\bar{\Psi}_h$, although these h independent columns are nonunique in the matrix $\bar{\Psi}_h$. There are always k dependent columns in the matrix $\bar{\Psi}_h$. From the mathematical point of view, one can adopt any numerical technique to eliminate k dependent columns from the matrix $\bar{\Psi}_h$ to obtain an equivalent higher-order modal matrix $\Psi_h \in R^{n,h}$. The term Φ_h can be substituted by Ψ_h to form PCMS $[\Phi_k, \Psi_h]$ (Ref. 18).

The equivalence between PCMS $[\Phi_k, \Psi_h]$ and the original complete modal space $[\Phi_k, \Phi_h]$ can be mathematically proven. The demonstration presented in Ref. 18 aims at the equivalence between the entire contribution of both Ψ_h and Φ_h. The previous mathematical proof is very powerful but not direct. In this Note, a new direct mathematical approach is presented to demonstrate the equivalence. The vector space theory normally used in the linear algebra has been adopted here.

Let us assume two vector subspaces as shown in Eqs. (4) and (5):

$$V_1 = \text{Span}(\Phi_k) = \text{Span}(\varphi_1, \varphi_2, \ldots, \varphi_k) \tag{4}$$

$$V_2 = \text{Span}(\Phi_h) = \text{Span}(\varphi_{k+1}, \varphi_{k+2}, \ldots, \varphi_n) \tag{5}$$

where $\Phi_k \in R^{n,k}$ and $\Phi_h \in R^{n,h}$. Obviously, V_1 and V_2 are mutually orthogonal complement subspaces with respect to matrices M and K, that is,

$$V = V_1 \oplus V_2 = \text{Span}(\Phi_k, \Phi_h), \qquad V_2 = V_1^{\perp} \tag{6}$$

in which $V_1^{\perp}$ represents the orthogonal complement space of V_1. The term V is a linear combination of the spaces with n dimensions. Let $\mathcal{P}$ be defined as a projection of V along V_1 at V_2; that is, if $\gamma \in V$, $\gamma = \alpha + \beta$, $\alpha \in V_1$, and $\beta \in V_2$, then

$$\mathcal{P}(\gamma) = \beta \tag{7}$$

When P indicates a projective matrix corresponding to $\mathcal{P}$, one should have

$$P[\Phi_k, \Phi_h] = [0, \Phi_h] \tag{8}$$

In addition, from an orthogonal relationship

$$[\Phi_k, \Phi_h]^T M [\Phi_k, \Phi_h] = I \tag{9}$$

one can get

$$[\Phi_k, \Phi_h]^{-1} = [\Phi_k, \Phi_h]^T M \tag{10}$$

Both the left- and right-hand sides of Eq. (8) are postmultiplied by $[\Phi_k, \Phi_h]^{-1}$ and $[\Phi_k, \Phi_h]^T M$, separately, so that the following expression can be obtained:

$$P = \Phi_h \Phi_h^T M = I - \Phi_k \Phi_k^T M = \bar{\Psi}_h \tag{11}$$

Let us define another subspace V_3 as

$$V_3 = \text{Span}(\bar{\Psi}_h) \tag{12}$$

Because the $\text{rank}(\bar{\Psi}_h) = h$, one knows $\dim V_3 = h = \dim V_2$. Thus, subspace V_3 is also an orthogonal complement subspace of V_1, that is,

$$V = V_1 \oplus V_3 = \text{Span}(\Phi_k, \Psi_h), \qquad V_3 = V_1^{\perp} \tag{13}$$

in which $\bar{\Psi}_h$ is replaced by Ψ_h. Comparing Eqs. (6) and (13) indicates that subspaces V_2 and V_3 are the same. One can conclude that PCMS $[\Phi_k, \Psi_h]$ and the original complete modal space $[\Phi_k, \Phi_h]$ are equivalent.

Orthogonalization Techniques

A. Triangular Decomposition Technique

Assume that $A \in R^{n,n}$ is a real symmetry matrix and $\text{rank}(A) = h < n$. Matrix A can be decomposed as

$$A = BB^T, \qquad B \in R^{n,n} \tag{14}$$

$$A = CC^T, \qquad C \in R^{n,h} \tag{15}$$

in which $\text{rank}(A) = \text{rank}(B) = \text{rank}(C) = h$. A triangular decomposition procedure for obtaining the matrices B and C was presented in Ref. 5.

From Eq. (2), the equivalent higher-order mode matrix Ψ_h also can be expressed as

$$\bar{\Psi}_h = \Phi_h \Phi_h^T = M^{-1} - \Phi_k \Phi_k^T \tag{16}$$

Because $\bar{\Psi}_h$ shown in Eq. (16) is a real symmetry matrix and $\text{rank}(\bar{\Psi}_h) = h < n$, $\bar{\Psi}_h$ can be decomposed as $\Psi_d \Psi_d^T$ (i.e., $\Psi_d = C$) obtained from Eqs. (14) and (15). Thus using Eq. (16) gives

$$\Psi_d \Psi_d^T = \Phi_h \Phi_h^T \tag{17}$$

The following orthogonality relationships can be proven:

$$\Psi_d^T M \Psi_d = I, \qquad \Psi_d^T M \Phi_k = 0 \tag{18}$$

Assume φ is an arbitrary column vector of Φ_k, and from Eq. (17) one knows

$$\Psi_d \Psi_d^T M \varphi = \Phi_h \Phi_h^T M \varphi = 0 \tag{19}$$

Let column vector $\bar{\varphi} = \Psi_d^T M \varphi$, and from Eq. (19) one gives

$$\Psi_d \bar{\varphi} = 0 \tag{20}$$

Because all column vectors of Ψ_d are linear independent, from Eq. (20) one obtains

$$\bar{\varphi} = \Psi_d^T M \varphi = 0 \tag{21}$$

That is, $\Psi_d^T M \Phi_k = 0$ exists. Moreover, using Eq. (17) can establish the following expression:

$$\Psi_d \Psi_d^T M \Psi_d \Psi_d^T = \Phi_h \Phi_h^T M \Phi_h \Phi_h^T = \Phi_h \Phi_h^T = \Psi_d \Psi_d^T \tag{22}$$

Therefore one knows

$$\Psi_d^T M \Psi_d = I = \mu_h \tag{23}$$

The aforementioned Ψ_d is a new equivalent higher-order mode matrix and is orthogonal with respect to M. However, Ψ_d still is not orthogonal to K. The just stated decomposition technique is efficient when the computation only needs the inversed matrix μ_h^{-1}.

B. Optimal Orthogonalizaqtion Method

In this section, the orthogonalization process of Ψ_h is presented based on an optimal Gram–Schmidt method.

1. Orthogonalization with Respect to M

Let $\tilde{\Psi}_h$ indicate the result generated after the orthogonalization of Ψ_h with respect to M. Define the following matrices with the variable number as the column of the matrices:

$$\Psi_h^{(j)} = [\psi_1, \psi_2, \ldots, \psi_j], \qquad \tilde{\Psi}_h^{(j)} = [\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_j] \tag{24}$$

Both Ψ_h and $\tilde{\Psi}_h$ are defined by Eq. (24) with $j = h$. Note that when $j = 1$, $\tilde{\Psi}_h^{(1)} = \tilde{\psi}_1 = \psi_1$; when $j = 2$, $\tilde{\Psi}_h^{(2)} = [\psi_1, \tilde{\psi}_2], \ldots$, etc.; that is, an arbitrary column of Ψ_h can be selected as the base vector of the orthogonalization process.

Let $(j + 1)$th orthogonalized vector be

$$\tilde{\psi}_{j+1} = \psi_{j+1} + \Delta\psi_{j+1} \tag{25}$$

Require

$$\tilde{\Psi}_h^{(j)T} M \tilde{\psi}_{j+1} = \tilde{\Psi}_h^{(j)T} M(\psi_{j+1} + \Delta\psi_{j+1}) = 0 \tag{26}$$

and

$$\varepsilon = \Delta\psi_{j+1}^T \Delta\psi_{j+1} = \min \tag{27}$$

The Lagrangian multiplier method is employed to solve the preceding extreme value problem with the constraint condition of Eq. (26). Lagrangian function is defined as

$$L = \varepsilon + 2\alpha^T \left[\tilde{\Psi}_h^{(j)T} M(\psi_{j+1} + \Delta\psi_{j+1})\right] \tag{28}$$

where $\alpha^T = (\alpha_1, \alpha_2, \ldots, \alpha_j)$ indicates a Lagrangian multiplier row vector. The following expression is found from $\delta L/\delta(\Delta\psi_{j+1}) = 0$:

$$\Delta\psi_{j+1} = -M\tilde{\Psi}_h^{(j)}\alpha \tag{29}$$

Embedding Eq. (29) into Eq. (26) gives

$$\alpha = \left[\tilde{\Psi}_h^{(j)T} M^2 \tilde{\Psi}_h^{(j)}\right]^{-1} \tilde{\Psi}_h^{(j)T} M \psi_{j+1} \tag{30}$$

Combining Eqs. (25), (29), and (30) yields

$$\tilde{\psi}_{j+1} = \psi_{j+1} - M\tilde{\Psi}_h^{(j)}\left[\tilde{\Psi}_h^{(j)T} M^2 \tilde{\Psi}_h^{(j)}\right]^{-1} \tilde{\Psi}_h^{(j)T} M \psi_{j+1} \tag{31}$$

in which the dimension of the inversed matrix $[\tilde{\Psi}_h^{(j)T} M^2 \tilde{\Psi}_h^{(j)}]^{-1}$ is large as j approaches $h - 1$. To save the computing time to find the inverse of the matrix, a procedure for solving the inverse of a partitioned matrix should be utilized such that the known $(j-1) \times (j-1)$ inversed matrix $[\tilde{\Psi}_h^{(j-1)T} M^2 \tilde{\Psi}_h^{(j-1)}]^{-1}$ can be used. This inversed matrix is already computed during the calculation of $\tilde{\psi}_j$. Clearly, the inversed matrix shown in Eq. (31) can be partitioned as shown in Eq. (32):

$$\left[\tilde{\Psi}_h^{(j)T} M^2 \tilde{\Psi}_h^{(j)}\right]^{-1} = \begin{bmatrix} \tilde{\Psi}_h^{(j-1)T} M^2 \tilde{\Psi}_h^{(j-1)} & \tilde{\Psi}_h^{(j-1)T} M^2 \tilde{\psi}_j \\ \tilde{\psi}_j^T M^2 \tilde{\Psi}_h^{(j-1)} & \tilde{\psi}_j^T M^2 \tilde{\psi}_j \end{bmatrix}^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \tag{32}$$

There are two different inversed matrices, $A_{11}^{-1} = [\tilde{\Psi}_h^{(j-1)T} M^2 \tilde{\Psi}_h^{(j-1)}]^{-1}$ and $(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$, for calculating the inverse of the partitioned matrix shown in Eq. (32). The former is already known, and the latter can be easily computed.

2. Orthogonalization with Respect to Both M and K

With the K-orthogonalization condition

$$\tilde{\tilde{\Psi}}_h^{(j)T} K(\psi_{j+1} + \Delta\psi_{j+1}) = 0 \tag{33}$$

Eq. (33) can be added to the condition extreme value problem shown in Eqs. (26) and (27), in which $\tilde{\tilde{\Psi}}_h$ denotes the result given after the orthogonalization of Ψ_h with respect to M and K. Similar procedures are adopted to yield

$$\tilde{\tilde{\psi}}_{j+1} = \psi_{j+1} - \left\{M\tilde{\tilde{\Psi}}_h^{(j)} - \left[K\tilde{\tilde{\Psi}}_h^{(j)} - M\tilde{\tilde{\Psi}}_h^{(j)} G_{MM}^{-1} G_{MK}\right] \times H_{KK}^{-1} G_{KM}\right\} G_{MM}^{-1} \tilde{\tilde{\Psi}}_h^{(j)T} M\psi_{j+1} - \left[K\tilde{\tilde{\Psi}}_h^{(j)} - M\tilde{\tilde{\Psi}}_h^{(j)} G_{MM}^{-1} G_{MK}\right] H_{KK}^{-1} \tilde{\tilde{\Psi}}_h^{(j)T} K\psi_{j+1} \tag{34}$$

in which

$$H_{KK} = G_{KK} - G_{KM} G_{MM}^{-1} G_{MK} \tag{35}$$

$$G_{MM} = \tilde{\tilde{\Psi}}_h^{(j)T} M^2 \tilde{\tilde{\Psi}}_h^{(j)}, \qquad G_{MK} = \tilde{\tilde{\Psi}}_h^{(j)T} MK \tilde{\tilde{\Psi}}_h^{(j)}$$
$$G_{KK} = \tilde{\tilde{\Psi}}_h^{(j)T} K^2 \tilde{\tilde{\Psi}}_h^{(j)}, \qquad G_{KM} = \tilde{\tilde{\Psi}}_h^{(j)T} KM \tilde{\tilde{\Psi}}_h^{(j)} \tag{36}$$

The term G_{MM}^{-1} can be computed quickly by using the preceding method for computing the inverse of the matrix. But computing H_{KK}^{-1} directly is tremendously time consuming. One can employ the following approximation to increase the computational efficiency:

$$H_{KK}^{-1} = \left(I - G_{KK}^{-1} G_{KM} G_{MM}^{-1} G_{MK}\right)^{-1} G_{KK}^{-1} \approx G_{KK}^{-1} + G_{KK}^{-1} G_{KM} G_{MM}^{-1} G_{MK} G_{KK}^{-1} \tag{37}$$

Thus, H_{KK}^{-1} can be computed quickly because the procedure of calculation of G_{MM}^{-1} and G_{KK}^{-1} is similar to that shown in Eq. (32).

After the orthogonalization, $\tilde{\tilde{\Psi}}_h$ or $\tilde{\Psi}_h$ are normalized with respect to M to obtain an improved complete modal space $[\Phi_k, \tilde{\tilde{\Psi}}_h]$ or $[\Phi_k, \tilde{\Psi}_h]$. The corresponding orthogonality conditions should be

$$[\Phi_k, \tilde{\tilde{\Psi}}_h]^T M [\Phi_k, \tilde{\tilde{\Psi}}_h] = I$$
$$[\Phi_k, \tilde{\tilde{\Psi}}_h]^T K [\Phi_k, \tilde{\tilde{\Psi}}_h] = \operatorname{diag}\begin{bmatrix} \Lambda_k & 0 \\ 0 & \mathcal{K} \end{bmatrix} \tag{38}$$

where $\mathcal{K} = \tilde{\tilde{\Psi}}_h^T K \tilde{\tilde{\Psi}}_h$ is a diagonal matrix.

Concluding Remarks

The PCMS theory has been investigated systematically and successfully applied to many technical fields. It is concluded that any techniques that can utilize the modal expansion procedure will make possible the use of the PCMS so that the accuracy of the results can be increased. To satisfy the precision requirement in different applications, the PCMS method has been further improved via an optimal Gram–Schmidt orthogonalization method and triangular decomposition technique.

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Integrating Experimental Data and Mathematical Models in Simulation of Physical Systems

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Nomenclature

A	= matrix with components $G(x_k, x_m)$
$B[\cdot]$	= boundary operator of mathematical model
$b\langle\cdot,\cdot\rangle$	= symmetric form associated with boundary operator B
c_m	= basis expansion coefficients
$\bar{c}$	= vector with components c_m
$E[\cdot]$	= mathematical expectation operator
F	= response of physical system
f	= arbitrary function exactly satisfying boundary operator B
$\tilde{f}$	= hypersurface that gives global minimum of Eq. (5)
f_a	= network response; numerical approximation of $\tilde{f}$
f_{cor}	= correction of f_0 using experimental data
$f_e(x_i)$	= experimental measurements of $F(x_i)$
$\bar{f}_e$	= vector with components $f_e(x_i)$
f_0	= response of mathematical model
$\bar{f}_0$	= vector with components $f_0(x_j)$
$G(\cdot,\cdot)$	= Green's function of mathematical model
g	= algebraic function
I	= identity matrix
$L[\cdot]$	= linear self-adjoined differential operator
$l\langle\cdot,\cdot\rangle$	= quadratic symmetric energy form associated with L
p	= weighting function
q	= Lagrange multiplier function
R^d	= d-dimensional Euclidean space
$R(\alpha)$	= residual of Eq. (20)
s	= parameter related to operator L, where $s > 0$
$\text{Var}[\cdot]$	= variance of a random variable
$x = (x_1, \ldots, x_d)$	= d-dimensional spatial coordinate
α	= regularization parameter where $\alpha \geq 0$
$\bar{\beta}$	= set of general artificial neural network parameters; set of basis function parameters and basis expansion coefficients
Δ_k	= radial basis function parameter
$\delta(\cdot)$	= Dirac delta function
δ_{ji}	= $\delta(x - x_j, x - x_i)$
$\epsilon(\cdot)$	= objective function
η_k	= radial basis function parameter
κ	= positive constant
$\Lambda[\cdot]$	= smoothness-based Tikhonov regularization functional
μ	= noise of measurement
ξ	= arbitrary variable
ρ	= positive constant
σ	= basis function
$\Phi(\cdot)$	= energy functional of mathematical model
ϕ, ψ	= arbitrary functions
Ω	= spatial domain of mathematical model
$\partial\Omega$	= spatial boundary of mathematical model
$\|\cdot\|_{H^s(\Omega)}$	= norm in Sobolev space of order s
$\langle\cdot,\cdot\rangle$	= inner product

I. Introduction

THE goal of engineering analysis is to obtain a comprehensive description of a physical system of interest. Three approaches to this goal exist: theoretical analysis of mathematical models, physical experimentation, and computational mechanics.[1] None of these approaches is superior, and they should be used in combination. Though it is known in the engineering community that successful analyses rest upon the proper balance of all three approaches, few attempts have been made in uniting experimental, theoretical, and numerical methods in the literature. It is the objective of this Note to develop a method that effectively combines all available information, from both experimental data and mathematical models, in the emulation of physical systems.

The technical approach utilizes a common tool borrowed from artificial neural network (ANN) applications, the theory of Tikhonov regularization of ill-posed problems.[2] Specifically, the problem of mathematical analysis of experimental data is treated as an ill-posed problem. Its regularization involves the introduction of additional information regarding the physical system. It is proposed that a priori mathematical models of physical systems be utilized at appropriate degrees of fidelity for regularization. The example problem is used to illustrate two major benefits of the approach: 1) to show the benefit of employing a priori mathematical models of low degree of fidelity for extrapolation from sparse data, and 2) to show the usefulness of incorporating a priori mathematical models in the processing of experimental data corrupted by noise. The radial basis expansion is used as the numerical approximation tool in the developed method.

II. Method for Data Analysis: Regularization by A Priori Mathematical Models

The popular Tikhonov regularization method[2] was originally adopted for ANN systems by Poggio and Girosi[3] and extensively used in ANN applications.[4,5] In this case, a set of constraints on the network approximation $f_a(x, \bar{\beta})$ is introduced through the Tikhonov regularization functional $\Lambda[f_a(x, \bar{\beta})]$, which specifies the penalty

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